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tion in the new variable must be a pure quadratic and the solution is readily completed. If we do not assume that the coefficients are real, one root may be real while the other is imaginary. In fact the roots may be the affixes of any two points.

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NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from January Number.]

PROPOSITION XXV. *If two straights* (*Fig.* 30.) AX, BX *existing in the same plane* (*standing upon* AB, *one indeed at an acute angle in the point* A, *and the other perpendicular at the point* B) *so always approach more to each other mutually, toward the parts of the point* X, *that nevertheless their distance is always greater than a certain assigned length, the hypothesis of acute angle is destroyed.*

PROOF. Let R be the assigned length. If therefore in BX is assumed a certain BK any chosen multiple of the proposed length R; it follows (from the preceding Scholion) that the perpendicular erected from the point K toward the parts of AX will meet it at some point L; and again (from the present hypothesis) it follows that this KL will be greater than the aforesaid length R. Furthermore BK is understood divided into portions KK, each equal to R, even until KB is itself equal to the length R. Finally from the points K are erected to BX perpendiculars meeting AX in points L, H, D, M, even to the point N nearest the point A. Now I proceed thus.

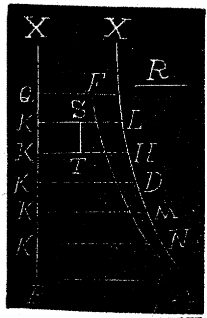

Fig. 30.

The four angles together of the quadrilateral $KHLK$, more remote from the base AB, will be (from the preceding Proposition) greater than the four angles together of the quadrilateral $KDHK$, nearer to this base; of which quadrilateral in the same way the four angles together will be greater than the four angles together of the quadrilateral $KMDK$ subsequent toward this base. And so always even to the last quadrilateral $KNAB$, whose four angles together assuredly will be the least, in reference to the four angles together of each of the quadrilaterals ascending toward the points X.

But since are present as many quadrilaterals described in the aforesaid manner, as are, except the base AB, perpendiculars let fall from points of AX to the straight *BX*; the sum of all the angles together, which are comprehended in these quadrilaterals can be reckoned. We assume that there are nine such perpendiculars let fall, and therefore so nine quadrilaterals.

We get (from Eu. I. 13) as equal to four rights the angles comprehended hither and yon at the two points of those eight perpendiculars, which lie in the middle between the base *AB* and the more remote perpendicular *LK*. So the sum of all these angles will be 32 rights.

There remain two angles at the perpendicular *LK*, and two at the base *AB*. But the angles one indeed at the point *K* and the other at the point *B* are supposed right; but the angle at the point *L* (from the Cor. after P. XXIII.) is obtuse. Wherefore (even neglecting the acute angle at the point A) the sum of all the angles which are comprehended by these nine quadrilaterals exceeds 35 rights. But hence follows, that the four angles together of the quadrilateral *KHLK*, more remote from the base lack less from four rights than the ninth part of one right; and that indeed even if an equal portion of the aforesaid sum of all the angles pertained to each of those quadrilaterals.

Therefore less yet will be the entered defect, since the sum of the four angles together of this quadrilateral *KHLK* was shown the greatest of all, in relation to the four angles together of the remaining quadrilaterals.

But again; in consequence of the supposition upon which this proposition proceeds, so great a length of *BK* can be assumed, that as many quadrilaterals as we choose may be made on bases *KK*, each equal to the assigned length *R*.

Wherefore the defect of the four angles together of this more remote quadrilateral *KHLK* from four rights is shown ever less both than a hundredth and than a thousandth, and thus under any assignable part of a right. Further however, *LK* and *HK* will be always (in accordance with the aforesaid supposition) greater than the designated length *R*. Therefore if in *KL* and *KH* are assumed *KS* and *KT* equal to *KK* or the length *R*; *ST* being joined, the two angles together *KST*, *KTS* will be greater, in hypothesis of acute angle, than the two angles together (from Cor. after P. XVI.) at the points *H* and *L* in the quadrilateral *THLS*, or the quadrilateral *KHLK*; and therefore (the common right angles at the points *K*, *K* being added) the four angles together of the quadrilateral *KTSK* will be greater than the four angles together of that quadrilateral *KHLK*.

But now, since on one hand is stable and given the quadrilateral *KTSK*, in as much as constant in the given base *KK*, which indeed is taken equal to the assigned length *R*, and again constant in the two perpendiculars *TK*, *SK* equal to this base, and finally in the joining *TS*, which comes out completely determinate; and on the other hand the four angles together of this stable and given quadrilateral have now been shown greater than the four angles together of the quadrilateral *KHLK* distant as far as we choose from the base *AB*; assuredly it follows, that the four angles together of this stable and given quadrilateral *KTSK* are greater than any sum of angles, which lacks however you choose of being four right angles; since already it has been shown that a quadrilateral *KHLK* can always be designated such that its four angles together shall fall short of four rights less than any assignable part of a right. Therefore the four angles together of this stable and given quadrilateral either are equal to four rights or greater.

But then (from P. XVI.) is established the hypothesis either of right angle or of obtuse angle ; and therefore (from P. V. and P. VI.) the hypothesis of acute angle is destroyed.

So is established that the hypothesis of acute angle will be destroyed, if two straights existing in the same plane so approach each other mutually ever more, that nevertheless their distance is always greater than any assigned length.

Hoc antem erat demonstrandum.

COROLLARY I. But (the hypothesis of acute angle destroyed) the controverted Pronunciatum Euclidaeum is manifest from P. 13 of this ; just as that by me in this place it would be disclosed I promised in Scholion III after P. XXI of this, where we spoke of the attempt of the Arab Nassaradin.

COROLLARY II. On the other hand from this proposition, and from the preceding XXIII is manifestly gathered that not sufficient for establishing Euclidean geometry are two following points. One is : that we designate by the name of parallels those straights, which existing in the same plane possess a common perpendicular. The second indeed ; that all straights existing in the same plane, of which there is no common perpendicular, and therefore which according to the assumed definition are not parallel, must, being produced toward either part ever more, somewhere meet each other, if not at a finite, at least at an infinite distance.

For again it would be requisite to demonstrate, that any two straights existing in the same plane, upon which a certain straight cutting makes two internal angles toward the same parts less than two right angles, nowhere else can receive a common perpendicular.

But that, this demonstrated, Euclidean geometry is most exactly established, will be shown below.

[To be Continued.]

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

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[Continued from January Number.]

XXVII. Let ABC be a triangle, right-angled at C. With O, the middle of AB, as a center, describe a circle to which either of the other sides, as BC, shall be tangent. Then,